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RESEARCH

Three-Loop Effect in r.m.s and Charge Radii of Heavy Flavored Mesons in a QCD Potential Model

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Abstract
We make an analysis of r.m.s. and charge radii of heavy flavored mesons introducing the three-loop correction to the static potential $V(r) = -\frac{4\alpha_s}{3r} + br$. Considering linear as parent and Coulomb as perturbation, the first order wave function is used to study the corresponding correction in coordinate space. The wave function is then used to study the r.m.s. and charge radii of heavy flavored mesons. The computed results are then compared with the data available in literature. The results show significant improvements compared to our earlier works.

Keywords Quantum chromodynamics · Dalgarno's method · R.m.s. radius · Charge radius · Three-loop effect

1 Introduction

The choice of Coulomb-plus-linear confinement potential, known as Cornell potential [1–3], $V(r) = -\frac{4\alpha_s}{3r} + br$ is one of the important ingredients of potential model. This potential has been used quite successful both in relativistic quark models [4–6] and non-relativistic quark models [7, 8]. For heavy mesons non-relativistic approach offers a satisfactory account to study the different properties such as mass spectrum, decay rates, charge radius, Isgur-Wise functions etc. [9]. Chen et al. [10] reported the spin average masses and root mean square radii of heavy mesons in a superstring theory inspired potential model. Recently, Omugbe et al. [11] also reported the approximate mass spectra and root mean square radii of quarkonia by using Cornell potential.

The same potential has been utilized to study the mass [12] as well as r.m.s. radii [13], charge radii [13], decay constants [14], form factors, slope and curvature of Isgur-Wise function [12] from time to time by us. Earlier [13] the potential was defined as $V(r) = -\frac{4\alpha_s}{3r} + br + c$, where 'c' is a constant which can shift the energy scale, but shouldn't effect the wave function of the system. But in perturbation method like Dalgarno's method of

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perturbation this authenticity is violated [13]. So, in current analysis, we remove the constant shift ‘c’ from the phenomenological potential.

In this work, we make an attempt to incorporate the three-loop effect in a specific potential model. The one-loop corrections to the potential were computed by Fischler and Billoire [15, 16]. Nearly twenty years later, in 1990’s, two-loop effect on the static potential has been reported in ref. [17–19]. The three-loop corrections to the potential have been reported more recently by Smirnov, Lee et al., in ref. [20, 21]. Recently, we have made an analysis of three-loop effect in Cornell potential to understand properties like masses of pseudoscalar mesons and decay constants [22, 23] using both Dalgarno’s method of perturbation [24] and Variationally Improved Perturbation Theory (VIPT) [25].

The present work reports the r.m.s. radii [26, 27] and charge radii [28] of mesons taking linear part as parent and Coulomb part of the potential as perturbation and incorporate the three-loop effect with improved strong coupling constant [22, 23].

The two fold motivation in this work are: (a) To check the three-loop effect on the approximate analytical solution of the mesons to a particular Potential Model and (b) To apply the bound state solution in obtaining the r.m.s. and charge radii of the mesonic system.

The present paper is organised as follows: In the next Section 2, the formalism is discussed for three-loop effect (Section 2.1) in QCD potential and the corresponding wave function (Section 2.2); in Section 2.3, we have defined r.m.s radius and charge radius and the relation between them respectively. Section 3 contains results for r.m.s. radii (Section 3.1) and charge radii (Section 3.2). Finally, Section 4 contains our conclusions.

2 Formalism

2.1 Three-Loop Effect

In momentum space, the static potential with three-loop correction takes the form [29],

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^2}\left[1 + \frac{\alpha_s(|\vec{q}|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^3\left(a_3 + 8\pi^2 C_A^3 ln\frac{\mu^2}{\vec{q}^2}\right) + ...\right]. \quad (1)$$

Here $C_A = N_c$ is the number of colors and $C_F = \frac{N_c^2-1}{2N_c}$ is the color factor and a_i’s are known as loop co-efficients. The term $ln\frac{\mu^2}{\vec{q}^2}$ represents the infrared divergences [30] which is a new feature of the three-loop correction to the potential and can be recovered with the help of (2) of ref. [20]. It is to be noted that the infrared divergence cancels in physical quantities when the ultrasoft gluons interaction is considered [20]. In order to suppress the infrared divergence, $\mu^2 = \vec{q}^2$ is chosen and the last term vanishes from (1).

After putting the numerical values of a_i’s in (1) [20],

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^2}[1 + \frac{\alpha_s}{\pi}(2.5833 - 0.2778n_f) + \left(\frac{\alpha_s}{\pi}\right)^2(28.5468 - 4.1471n_f + 0.0772n_f^2) + \left(\frac{\alpha_s}{\pi}\right)^3(209.884 - 51.4048n_f + 2.9061n_f^2 - 0.0214n_f^3)]. \quad (2)$$

In coordinate space, the corresponding equation can be written as [23],

$$V(r) = -\frac{C_F \alpha_s(\mu')}{r}[1 + \frac{\alpha_s}{\pi}(2.5833 - 0.2778 n_f) + \left(\frac{\alpha_s}{\pi}\right)^2 (28.5468 - 4.1471 n_f + 0.0772 n_f^2) + \left(\frac{\alpha_s}{\pi}\right)^3 (209.884 - 51.4048 n_f + 2.9061 n_f^2 - 0.0214 n_f^3)]. \quad (3)$$

The relationship between improved three-loop strong coupling constant and standard leading order strong coupling constant is further given by the relation [23]

$$\alpha_V(\frac{1}{r}) = \alpha_s(\mu')[1 + \frac{\alpha_s}{\pi}(2.5833 - 0.2778 n_f) + \left(\frac{\alpha_s}{\pi}\right)^2 (28.5468 - 4.1471 n_f + 0.0772 n_f^2) + \left(\frac{\alpha_s}{\pi}\right)^3 (209.884 - 51.4048 n_f + 2.9061 n_f^2 - 0.0214 n_f^3)]. \quad (4)$$

The values of effective strong coupling constant $\alpha_V(\frac{1}{r})$ for one-, two-, three-loop effect are calculated in ref. [22] as

Here, it is to be mentioned that the values of μ' is considered to be the mass of heavy quarks respectively for charmonium and bottomonium scales, since for heavy-light mesons the mass of the heavy quark is dominant over the light quark mass.

2.2 The QCD Potential and the Corresponding Wave Function

We calculate the total wave function using Dalgarno's method of perturbation for the Cornell potential,

$$V(r) = -\frac{4\alpha_s}{3r} + br, \quad (5)$$

where the short-range gluon exchange interaction between a quark-antiquark is represented by Coulomb term and the large-scale quark confinement is represented by linear term.

In general, the linear confining potential is expected to be more dominant than the Coulombic part. Taking the Coulomb part as parent and linear as perturbation, the perturbation is possible only for very small value of 'b', such as in the infinite mass limit $b \leq 0.03 GeV^2$ [31], which is very much less than the standard 'b' value ($\approx 0.183 GeV^2$) of charmonium spectroscopy. Hence, Coulombic part is considered to be perturbed in this study[1].

The wave function with $H_0 = -\frac{\nabla^2}{2\mu} + br$ as parent and $H' = -\frac{4\alpha_V}{3r}$ as perturbation is given by [14].

$$\psi(r) = \frac{N'}{r}\left[1 + A_0 r^0 + A_1(r) r + A_2(r) r^2 + A_3(r) r^3 + A_4(r) r^4 + ...\right] A_i[\rho_1 r + \rho_0], \quad (6)$$

where $A_i[r]$ is the Airy function [33] and N' is the normalization constant,

$$N' = [\int_0^{r_0} 4\pi [1 + A_0 r^0 + A_1(r) r + A_2(r) r^2 + A_3(r) r^3 + A_4(r) r^4 + ...]^2 (A_i[\rho_1 r + \rho_0])^2 dr]^{-\frac{1}{2}}. \quad (7)$$

Even though the Airy's function vanishes exponentially as $r \to \infty$ [33] and is normalizable too, a cut off r_0 is used as the upper limit of integration in the perturbative procedure.

[1] In Appendix A we have also shown the results for r.m.s. and charge radii considering the wave function with Coulombic part as parent and linear as perturbation [32].

Further it is found that this scale is independent of the property of the Airy's function and it is sensitive to r.m.s. radius and charge radius.

The co-efficients A_0, A_1, A_2, A_3, A_4 etc. of the series solution as occured in Dalgarno's method of perturbation, are the functions of α_V, μ and b:

$$A_0 = 0, \tag{8}$$

$$A_1 = \frac{-2\mu \frac{4\alpha_V}{3}}{2\rho_1 k_1 + \rho_1^2 k_2}, \tag{9}$$

$$A_2 = \frac{-2\mu W^1}{2 + 4\rho_1 k_1 + \rho_1^2 k_2}, \tag{10}$$

$$A_3 = \frac{-2\mu W^0 A_1}{6 + 6\rho_1 k_1 + \rho_1^2 k_2}, \tag{11}$$

$$A_4 = \frac{-2\mu W^0 A_2 + 2\mu b A_1}{12 + 8\rho_1 k_1 + \rho_1^2 k_2} \tag{12}$$

and so on.

The different parameters in the potential model are given by,

$$\rho_1 = (2\mu b)^{\frac{1}{3}} \tag{13}$$

$$\rho_0 = -\left[\frac{3\pi(4n-1)}{8}\right]^{\frac{2}{3}} \tag{14}$$

(In our case n=1 for ground state)

$$k_1 = 1 + \frac{k}{r} \tag{15}$$

$$k = \frac{0.355 - (0.258)\rho_0}{(0.258)\rho_1} \tag{16}$$

$$k_2 = \frac{k^2}{r^2} \tag{17}$$

$$W^1 = \int \psi^{(0)\star} H' \psi^{(0)} d\tau \tag{18}$$

$$W^0 = \int \psi^{(0)\star} H_0 \psi^{(0)} d\tau. \tag{19}$$

2.3 R.m.s. and Charge Radii

The r.m.s. radius of the bound state of quark and anti quark like meson can be defined in terms of wave function as [26, 27]

$$\langle r_{rms}^2 \rangle = \int_0^{r_0} r^2 [\psi(r)]^2 dr \tag{20}$$

with radial wave function $\psi(r)$ as defined in (6).

The average charge radii square [28] for the meson can be extracted from the form factors at their low Q^2 behaviour using the relation,

$$\langle r_E^2 \rangle = -6 \frac{d^2}{dQ^2} eF(Q^2)|_{Q^2=0}, \tag{21}$$

where the elastic charge radii form factor for a charged system of point quarks has the form

$$F(Q^2) = \sum_{i=1}^{2} \frac{e_i}{Q_i} \int_0^{r_0} 4\pi r \mid \psi(r) \mid^2 sin(Q_i r) dr, \tag{22}$$

where Q^2 is the four momentum transfer square and e_i is the charge of the i^{th} quark/antiquark with

$$Q_i = \frac{\sum_{j \neq i} m_j Q}{\sum_{i=1}^{2} m_i}, \tag{23}$$

where Q_i describes how the virtuality Q^2 is shared between the quark and antiquark pair of the meson and m_i and m_j are the masses of the i^{th} and j^{th} quark/antiquark respectively.

R.m.s radii of mesons are of great interest for understanding the property concerning the average size of the bound state $< r^2 >$ of the quark wave function of the meson. On the other hand, the mean square charge radii of mesons are the deviation from the centre of mass co-ordinate squared weighted by the quark and antiquark constituents of the meson,

$$\langle r^2 \rangle = \frac{(Q_Q m_{\bar{q}}^2 + Q_{\bar{q}} m_Q^2)\langle \delta^2 \rangle}{(m_Q + m_{\bar{q}})^2},$$

where Q_Q and $Q_{\bar{q}}$ are charge of the quark and anti-quark and m_Q and $m_{\bar{q}}$ mass of quark and anti-quark respectively.

$\delta = r_Q - r_{\bar{q}}$ is the relative coordinate.

A simple approximate relationship between the r.m.s. and charge radii is obtained earlier in ref. [13].

As derived by Godfrey and Isgur [35] the relationship between the two can be found from the following equation

$$r_E^2 = \sum_i e_i \left[< {r_i}^2 > + \frac{3}{4m_i^2} \int d^3 p \mid \Phi(p) \mid^2 \left(\frac{m_i}{E_i} \right)^{2f} \right], \tag{24}$$

e_i is the charge of the i^{th} quark/antiquark, $\Phi(p)$ is the quark momentum distribution and the exponent f can be determined in a semi-empirical way (Table 1).

Table 1 Values of $\alpha_V(\frac{1}{r})$

n_f	$\alpha_s(\mu')$	LO	NLO	NNLO	NNNLO
4	0.39	0.450	0.544	0.670	0.730
5	0.22	0.259	0.280	0.297	0.303

Table 2 Three-loop effect on r.m.s. radii with different r_0 values

Meson	r_{rms} (Fermi)							
	$r_0 =$ 0.197 Fermi	$r_0 =$ 0.394 Fermi	$r_0 =$ 0.591 Fermi	$r_0 =$ 0.788 Fermi	Ref. [9]	Ref. [36]	Ref. [28]	Ref.[34]
$D(c\bar{u}/c\bar{d})$	0.38515	0.13468	0.07740	0.05840	–	–	–	–
$D_s(c\bar{s})$	0.38321	0.12967	0.07740	0.05837	–	–	–	–
J/$\psi(c\bar{c})$	0.38149	0.12232	0.07728	0.05825	0.4490	0.4453	0.4839	0.37
$B_c(\bar{b}c)$	0.42125	0.11856	0.07749	0.05829	–	–	–	
$\tau(b\bar{b})$	0.41425	0.11881	0.07711	0.05804	0.2249	0.2211	0.2671	0.22

Table 3 R.m.s. radii for J/$\psi(c\bar{c})$ with $\alpha_V(\frac{1}{r})$ values

With α_V for	$r_{r.m.s.}$ (Fermi)
NNNLO	0.4562
NNLO	0.4582
NLO	0.4640
LO	0.4711
Ref. [9]	0.4490
Ref. [36]	0.4453
Ref. [28]	0.4839
Ref. [34]	0.37

3 Results

Using (20) and (21), with the wave function given by (6), we compute the r.m.s. and charge radii of heavy flavored mesons. The corresponding results are tabulated in Tables 2, 3, and 4 and compared with the available data.

The input parameters in the numerical calculations are the same as used in our previous work [12–14]; $m_u = 0.336GeV$, $m_d = 0.336GeV$, $m_s = 0.483GeV$, $m_c = 1.55GeV$ and $m_b = 4.95GeV$, b=0.183 GeV^2.

3.1 Results for r.m.s. Radii with Three-Loop Effect

The sensitivity of the r.m.s radii of the mesons including J/$\psi(c\bar{c})$ and $\tau(b\bar{b})$ mesons are presented for different values of r_0 in Table 2 and compared with the available data. We have used *Mathematica 7* for the numerical calculations.

The analysis shows that with the rising values of r_0, the r.m.s. radii of mesons decreases and it is evident from Table 2 that for J/$\psi(c\bar{c})$ meson the results are in agreement for

Table 4 R.m.s. radii for $\tau(b\bar{b})$ with $\alpha_V(\frac{1}{r})$ values

With α_V for	$r_{r.m.s.}$ (Fermi)
NNNLO	0.2493
NNLO	0.2479
NLO	0.2436
LO	0.2370
Ref. [9]	0.2249
Ref. [36]	0.2211
Ref. [28]	0.2671
Ref. [34]	0.22

Table 5 The sensitivity of charge radii of the mesons with different r_0 values

Meson	$< r^2 > (Fermi^2)$								
	Present Work				Previous Work[14]	Previous Work [37]	Previous Work [38]	Ref. [28]	Ref. [36]
	$r_0 =$ 0.197 Fermi	$r_0 =$ 0.394 Fermi	$r_0 =$ 0.591 Fermi	$r_0 =$ 0.788 Fermi					
$D^+(c\bar{d})$	0.0106	0.0374	0.0893	0.1730	0.265	0.134	0.011	0.184	0.219
$D^0(c\bar{u})$	-0.0184	-0.0652	-0.1556	-0.3015	-0.463	-0.234	-0.013	-0.304	-0.403
$D_s^+(c\bar{s})$	0.0098	0.0334	0.0828	0.1549	0.222	0.126	0.010	0.124	–
$B^+(u\bar{b})$	0.0257	0.0955	0.2076	0.3634	0.538	2.96	0.060	0.378	–
$B^0(d\bar{b})$	-0.0127	-0.0474	-0.1031	-0.1804	-0.267	-1.47	-0.030	-0.187	–
$B_s^0(s\bar{b})$	-0.0119	-0.0432	-0.0938	-0.1571	-0.215	-1.37	-0.025	-0.119	–

Table 6 Charge radii of the mesons with $r_0 = 4GeV^{-1} = 0.788Fermi$ for 3-loop, 2-loop, 1-loop and LO

Meson	$< r^2 > (Fermi^2)$					
	NNNLO	NNLO	NLO	LO	Ref. [28]	Ref. [36]
$D^+(c\bar{d})$	0.1730	0.1711	0.1666	0.1628	0.184	0.219
$D^0(c\bar{u})$	-0.3015	-0.2981	-0.2903	-0.2836	-0.304	-0.403
$D_s^+(c\bar{s})$	0.1549	0.1531	0.1489	0.1454	0.124	–
$B^+(u\bar{b})$	0.3634	0.3626	0.3605	0.3579	0.378	–
$B^0(d\bar{b})$	-0.1804	-0.1800	-0.1790	-0.1777	-0.187	–
$B_s^0(s\bar{b})$	-0.1571	-0.1568	-0.15593	-0.1547	-0.119	–

$r_0 < 0.197Fermi$ ($r_0 < 1GeV^{-1}$) and for $\tau(b\bar{b})$ the results are achievable for a range of $0.197Fermi < r_0 < 0.394Fermi$ ($1GeV^{-1} < r_0 < 2GeV^{-1}$).[2]

Following the results of Table 2, let us see how r.m.s. radii vary with different α_V values of Table 1 in three-loop, two-loop, one-loop and in LO with $r_0 < 1GeV^{-1}$ (say, $r_0 = 0.8GeV^{-1} = 0.157Fermi$) for J/$\psi(c\bar{c})$ and with $1GeV^{-1} < r_0 < 2GeV^{-1}$ (say, $r_0 = 1.6GeV^{-1} = 0.315Fermi$) for $\tau(b\bar{b})$.

As can be seen from Tables 3 and 4, the results of r.m.s. radii with three-loop effect are close to the available data.

3.2 Results for Charge Radii with Three-Loop Effect

In the study of r.m.s radii we have seen that the cut off r_0 is different for charmonium and bottomonium scale. To check the sensitivity of the scale we compute the charge radii for heavy-light mesons by using (21) in Table 5 and compared with our previous work [14, 37, 38] and with the values of others [28, 36].

Table 5 clearly shows that charge radii values decrease, if we decrease the value of r_0 unlike in the case of r.m.s. radii. The sensitivity results indicate that charge radii results are better for $r_0 \geq 0.788$ Fermi ($r_0 \geq 4GeV^{-1}$). However, for the particular value of $r_0 = 4GeV^{-1} = 0.788Fermi$, the results for D and B mesons are calculated with different α_V values (Table 1) found to be comparable with the available data as is shown in Table 6.

Further, it is to be mentioned that J/$\psi(c\bar{c})$ and $\tau(b\bar{b})$ mesons don't have charge radii as they are charge neutral particles.

4 Conclusions

In this article, we have introduced the three-loop correction to the static potential and obtained the strong coupling constant in charmonium and bottomonium scale. The results have been compared with leading order, one-loop, three-loop effects in Tables 3, 4, and 6. The three-loop effect is further studied in computing r.m.s. and charge radii of heavy-light mesons. In the study, a cut off r_0 is also introduced in the model and sensitivity of the same is checked for heavy flavored mesons. The three-loop effect in calculation of strong coupling constant in V-scheme (α_V) indicates two different cut off r_0; for charmonium and bottomonium scale. We see for $r_0 = 0.8GeV^{-1} = 0.157$ Fermi, the r.m.s. radii for charmonium is found to be

[2] $5.076GeV^{-1} = 1Fermi$

comparable with other available data, whereas for bottomoium the results are comparable with available data for cut off $r_0 = 1.6GeV^{-1} = 0.315$ Fermi. In the analysis, we find that the three-loop correction to the potential model provides an improved result for r.m.s. and charge radii of heavy flavored mesons with a short range of cut off scale r_0. Further this scale is found to be approximately matched with the constraint parameterisation range of Cornell potential $0.2Fermi < r < 1Fermi$ [39].

A Appendix

R.m.s. Radii with Coulomb Parent and Linear Parturbed Wave Function [12, 32]

In previous works [12–14, 32], the relativistic effect was incorporated by introducing a Dirac modification factor $(\frac{r}{a_0})^{-\epsilon}$ to the wave function, where $\epsilon = 1 - \sqrt{1 - \left(\frac{4}{3}\alpha_s\right)^2}$. Later as discussed in ref. [40], due to this factor, the wave function develops a singularity, when $r \to 0$ and scalar meson masses become negative. We therefore disregard this factor in the present analysis and considered the total wave function as

$$\psi^{total}(r) = \frac{N}{\sqrt{\pi a_0^3}} \left[1 - \frac{1}{2}\mu b a_0 r^2\right] e^{-\frac{r}{a_0}}, \tag{A.1}$$

where N is the normalization constant and

$$a_0 = \left(\frac{4}{3}\mu\alpha_V\right)^{-1}.$$

Using (A.1) in (20), we obtain Tables 7 and 8.

Table 7 R.m.s. radii for $J/\psi(c\bar{c})$ with $r_0 = 0.8GeV^{-1} = 0.157Fermi$ with different loop effect

With α_V for	$r_{r.m.s.}$ (Fermi)
NNNLO	0.5016
NNLO	0.4943
NLO	0.4795
LO	0.4690
Ref. [9]	0.4490
Ref. [36]	0.4453
Ref. [28]	0.4839

Table 8 R.m.s. radii for $\tau(b\bar{b})$ with $r_0 = 1.6GeV^{-1} = 0.315Fermi$ with different loop effect

With α_V for	$r_{r.m.s.}$ (Fermi)
NNNLO	0.3130
NNLO	0.3095
NLO	0.2998
LO	0.2884
Ref. [9]	0.2249
Ref. [36]	0.2211
Ref. [28]	0.2671

Charge Radii with Coulomb Parent and Linear Parturbed Wave Function [12, 32]

Using (A.1) in (21) we obtain the following table.

Table 9 Charge radii of the mesons with $r_0 = 4GeV^{-1} = 0.788Fermi$ for 3-loop, 2-loop, 1-loop and LO

Meson	$< r^2 > (Fermi^2)$					
	NNNLO	NNLO	NLO	LO	Ref. [28]	Ref. [36]
$D^+(c\bar{d})$	0.0762	0.0947	0.1674	0.2186	0.184	0.219
$D^0(c\bar{u})$	-0.1328	-0.1650	-0.2916	-0.3809	-0.304	-0.403
$D_s^+(c\bar{s})$	0.0618	0.0754	0.1398	0.1954	0.124	–
$B^+(u\bar{b})$	0.5826	0.5832	0.5843	0.5847	0.378	–
$B^0(d\bar{b})$	-0.1309	-0.1345	-0.2901	-0.2903	-0.187	–
$B_s^0(s\bar{b})$	-0.2691	-0.2695	-0.2705	-0.2712	-0.119	–

It is evident from Tables 7, 8, and 9 that results for r.m.s. and charge radii are better for the wave function with linear as parent and Coulomb as perturbation [14] than one with the Coulomb as parent and linear as perturbation [32].

Author Contributions The corresponding author TD did all the calculations and wrote the original draft. KKP carried out editing . KKP and DKC carried out the discussion of results. All the authors reviewed the manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no competing interests.

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